Fluctuation scaling in complex systems

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Outline

* Taylor's law a.k.a. Fluctuation scaling
* Empirical data and "theory" in parallel
    * Random walks
    * Forests
    * Coins
    * Humans

Taylor's law or fluctuation scaling

NATURE March 4, 1961 VOL. 189

AGGREGATION, VARIANCE AND THE MEAN

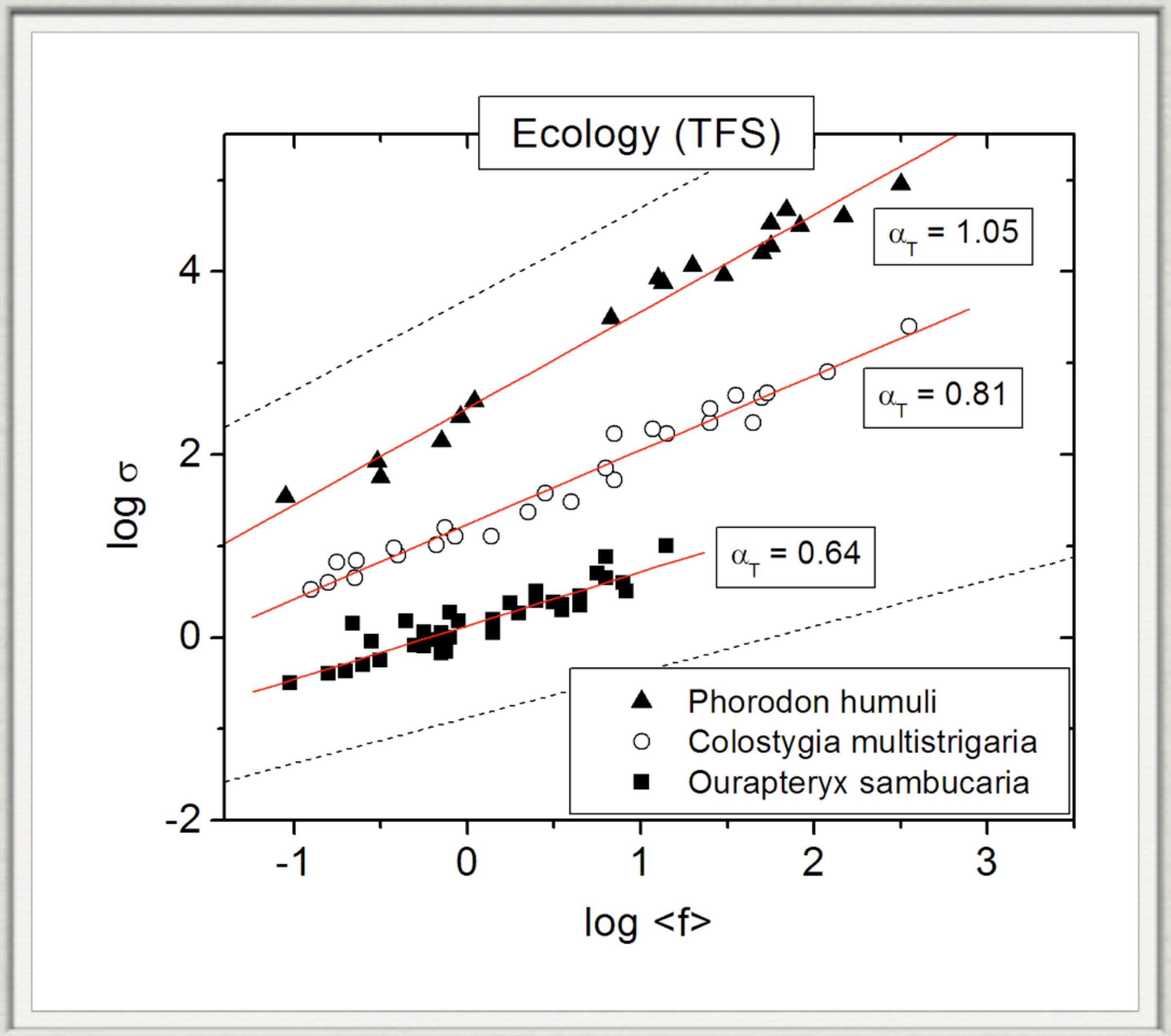
By L. R. TAYLOR

Department of Entomology, Rothamsted Experimental Station, Harpenden, Herts

(Temporal) Fluctuation scaling

- Take (stable) populations i of some species, and observe them in time
- Calculate the mean and the variation of the specimen count
- Plot the two → $\langle f_i \rangle$ → σ_i

Fluctuation scaling

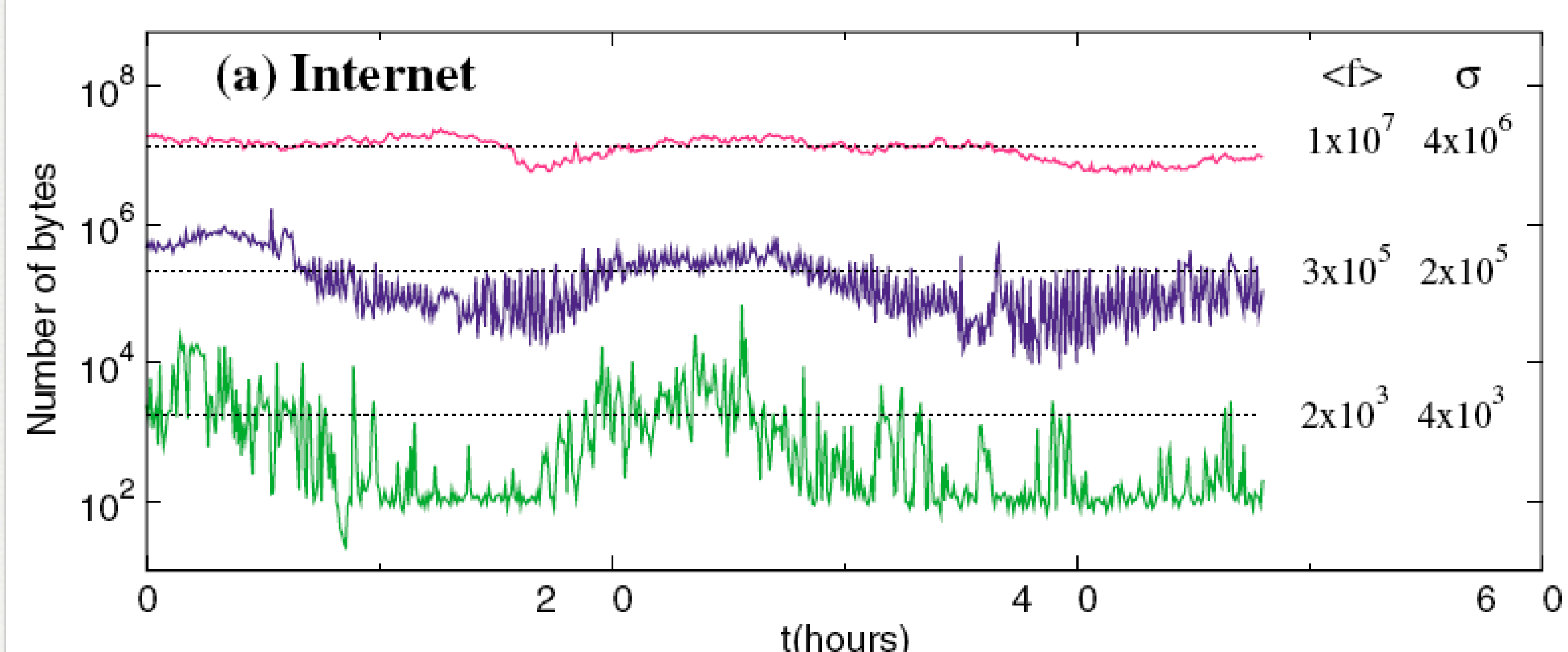


$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

VOLUME 92, NUMBER 2 PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2004

Fluctuations in Network Dynamics

M. Argollo de Menezes and A.-L. Barabási
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556, USA
(Received 11 June 2003; published 13 January 2004)



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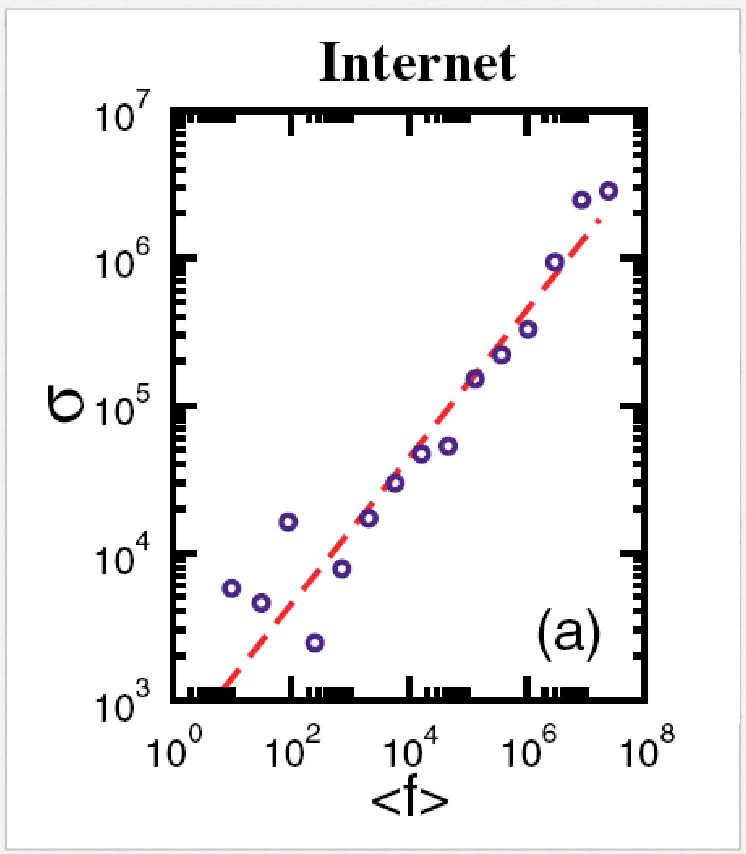

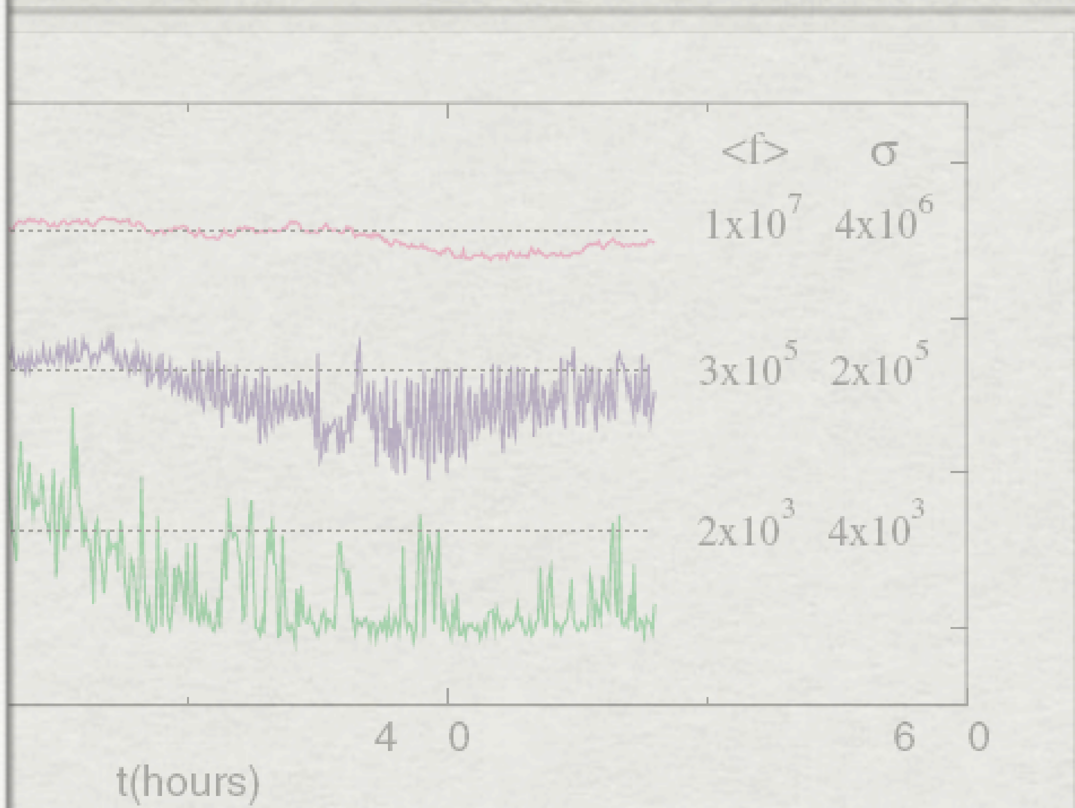

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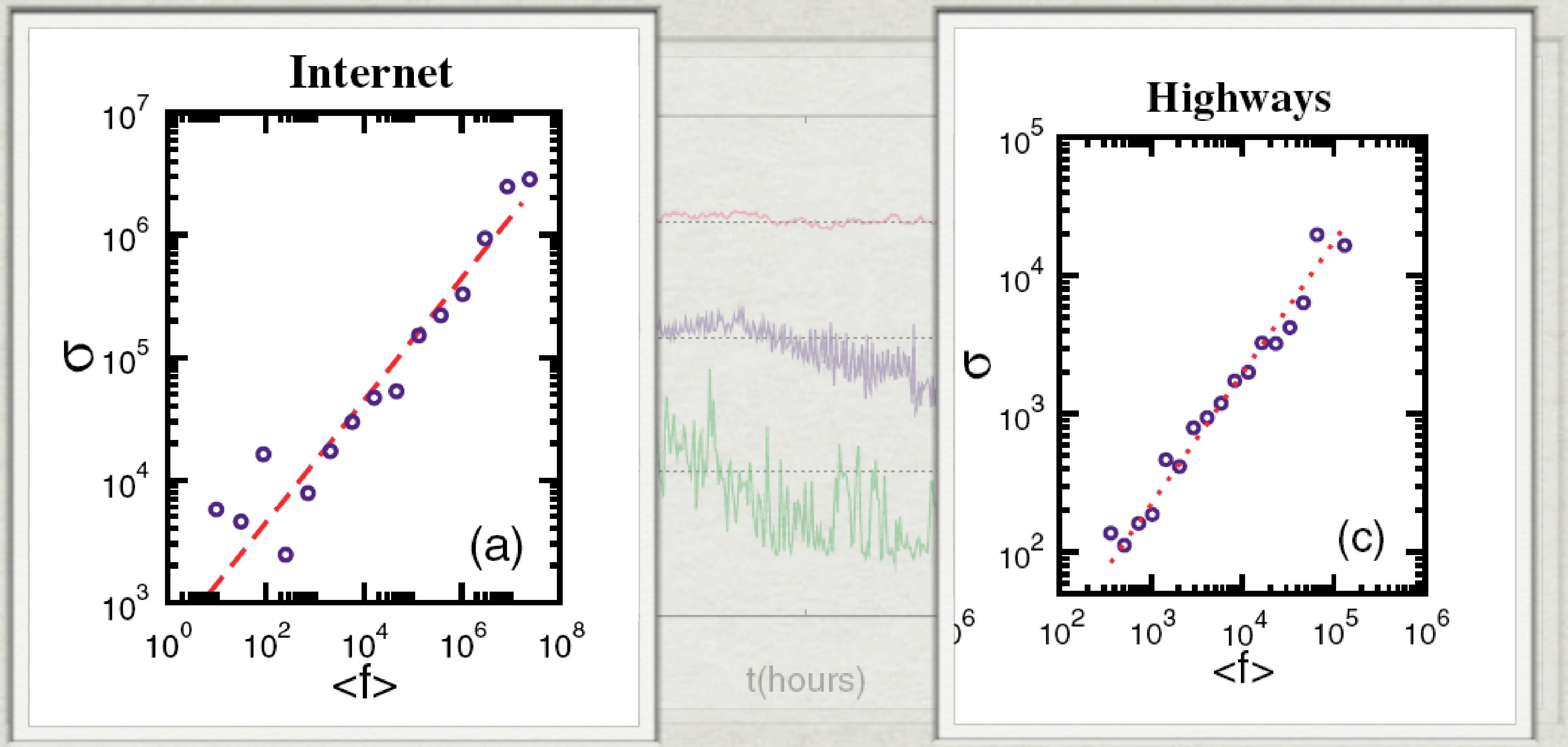


Fluctuation scaling

- Take similar systems i, and observe them in time
- Calculate the mean and the variation of a positive additive signal
- Then vary i → $\langle f_i \rangle$, σ_i → $\sigma_i \propto \langle f_i \rangle^{\alpha}$

Subj.	System	T/E	Refs.
Networks	Random walk	T	[7, 31, 33]
	Network models	T	[34, 35]
	Highway network	T	[7, 31]
	World Wide Web	T	[7, 31]
	Internet	T	[7, 31, 32]
Phy.	Heavy ion collisions	E	[26–28]
	Cosmic rays	E	[29, 30]
Soc./econ.	Stock market	T	[8, 56, 57, 60]
	Stock market	E	this review
	Business firm growth rates	E	[62, 63]
	Email traffic	T	this review
	Printing activity	T	this review
Cl.	River flow	T	[64, 65]
	Precipitation	T	[66]
Ecology/pop. dyn.	Forest reproductive rates	T	[46, 47]
	Satake-Iwasa forest model	T	[45]
	Crop yield	T	[6]
	Animal populations	T, E	[5, 10, 15, 16]
	Diffusion Limited population	E	[17]
	Population growth	T	[67, 68]
	Exponential dispersion models	E	[18, 21, 69]
	Interacting population model	T	[37]
Life sciences	Cell numbers	E	[20]
	Protein expression	T	[55]
	Gene expression	T	[70, 71]
	Individual health	E	[72]
	Tumor cells	E	[21]
	Human genome	E	[22, 23]
	Blood flow	E	[69]
	Oncology	E	[21]
	Epidemiology	T	[53, 54]

Table II: A list of some studies where fluctuation scaling/Taylor's law was directly applied or implied by a similar formalism. Groups were assigned by subject areas, Phy. = Physics, Cl. = Climatology. The column T/E shows the type of fluctuation scaling, T: temporal, E: ensemble.

Fluctuation scaling

- Take similar systems, and observe them in time
- Calculate the mean and the variation of some positive signal
- Then vary i

$$\langle f_i \rangle$$

$$\sigma_i$$

$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

Why do we care?

- the value of α varies mostly in [1/2, 1]

$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

Why do we care?

- the value of α varies mostly in [1/2, 1]
- simple dynamical rules?

$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

Why do we care?

- the value of α varies mostly in [1/2, 1]
- simple dynamical rules?
- it is NOT a universal exponent

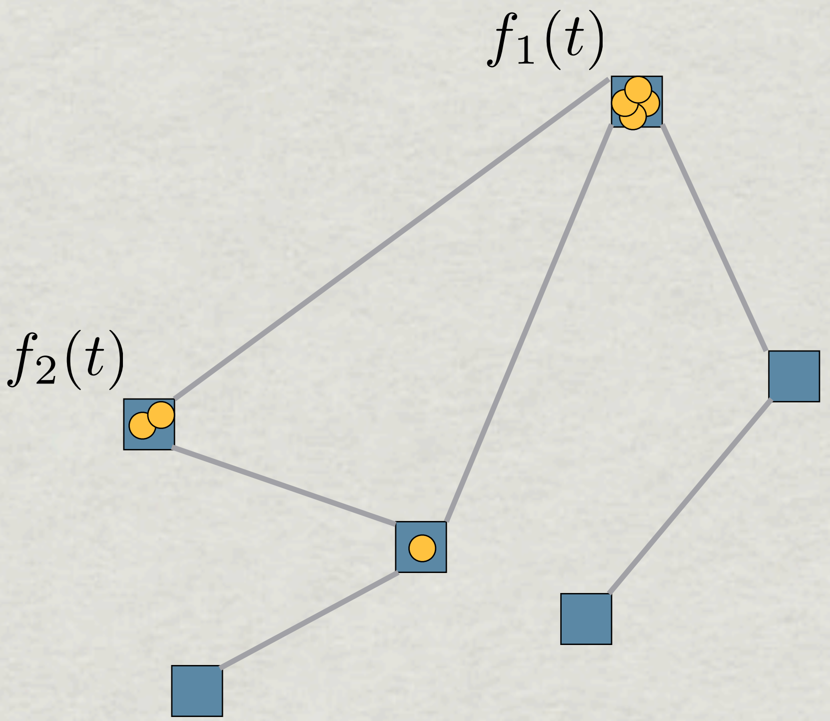
$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

The possible values of α

- Random walks
- Forests
- Coins (?)
- Humans

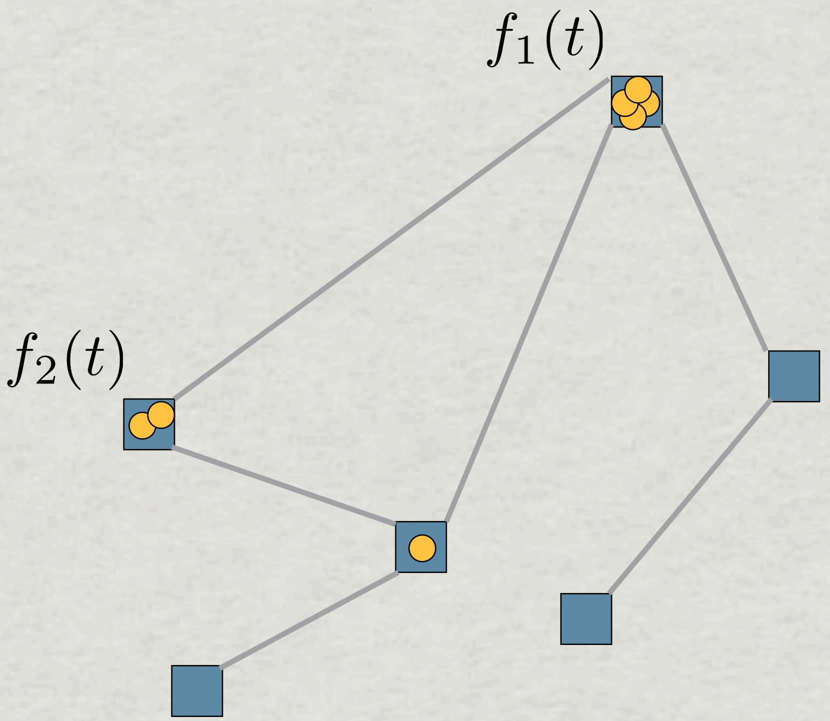
Random walks

- N walkers (many)



Random walks

- N walkers (many)
- $V_{n,i}(t) = \begin{cases} 1 & \text{if walker } n \text{ is on} \\ & \text{node } i \text{ at time } t, \\ 0 & \text{if not.} \end{cases}$



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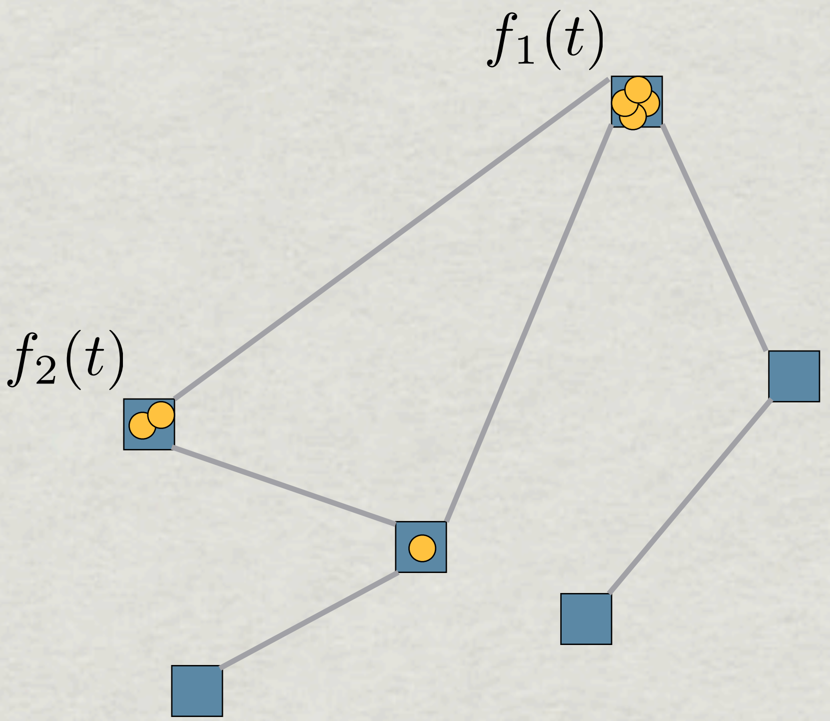
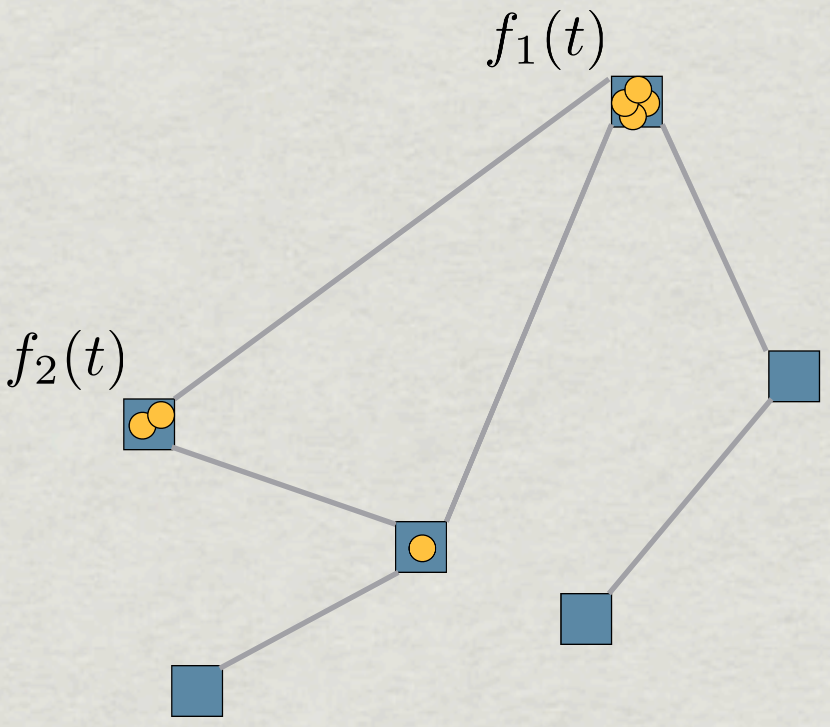


Random walks



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- $\langle f_i \rangle = N \langle V_{n,i} \rangle = N p_i \propto k_i$

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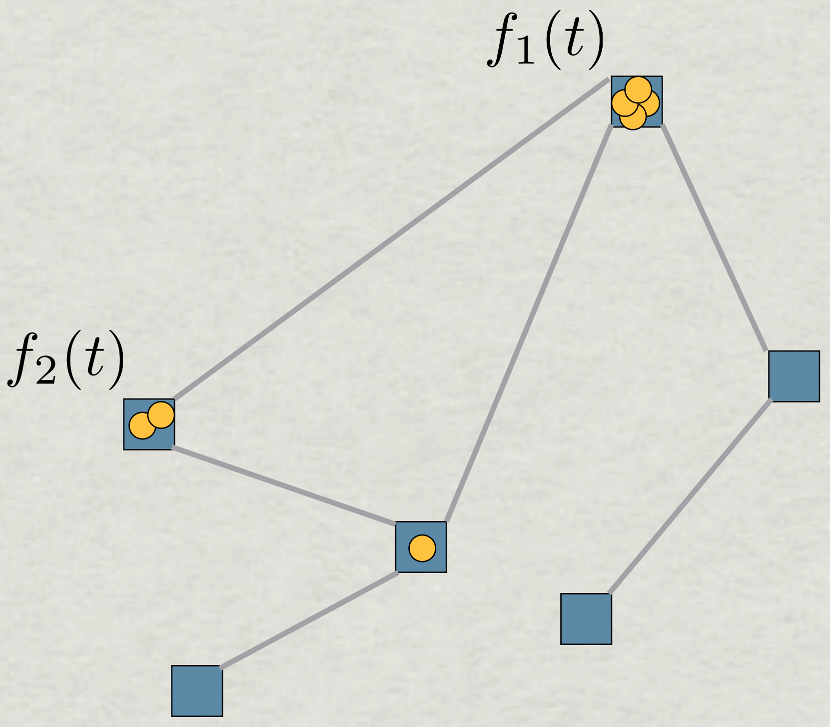
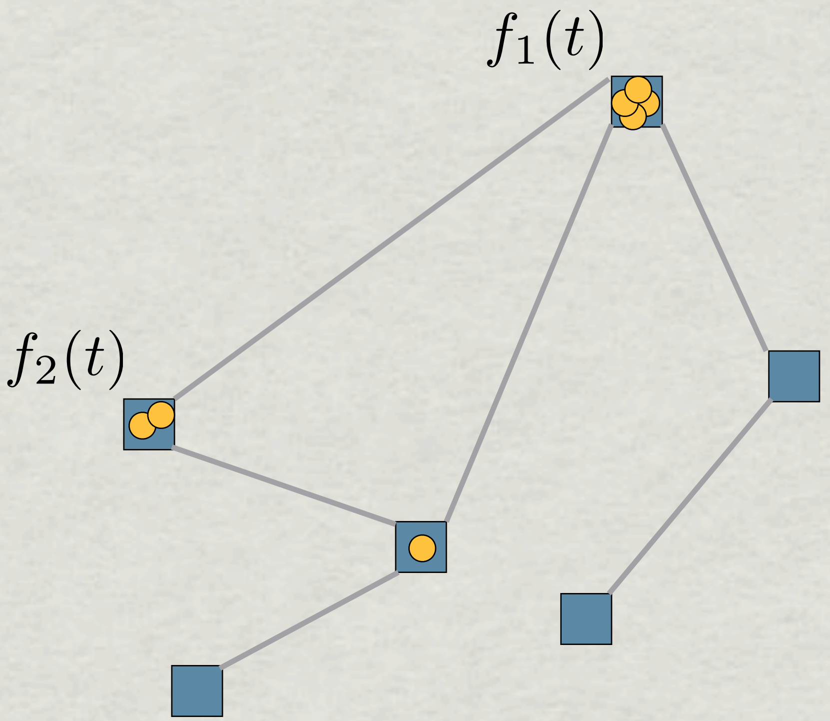
- $\sigma_i^2 = N p_i$



Random walks



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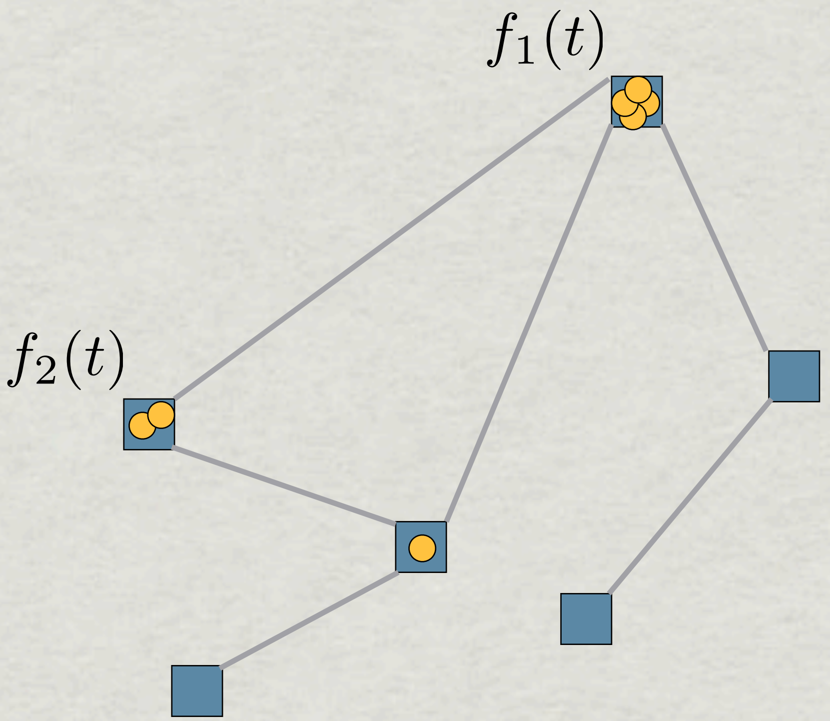
- $\sigma_i^2 = N p_i$

$\alpha = 1/2$

Random walks

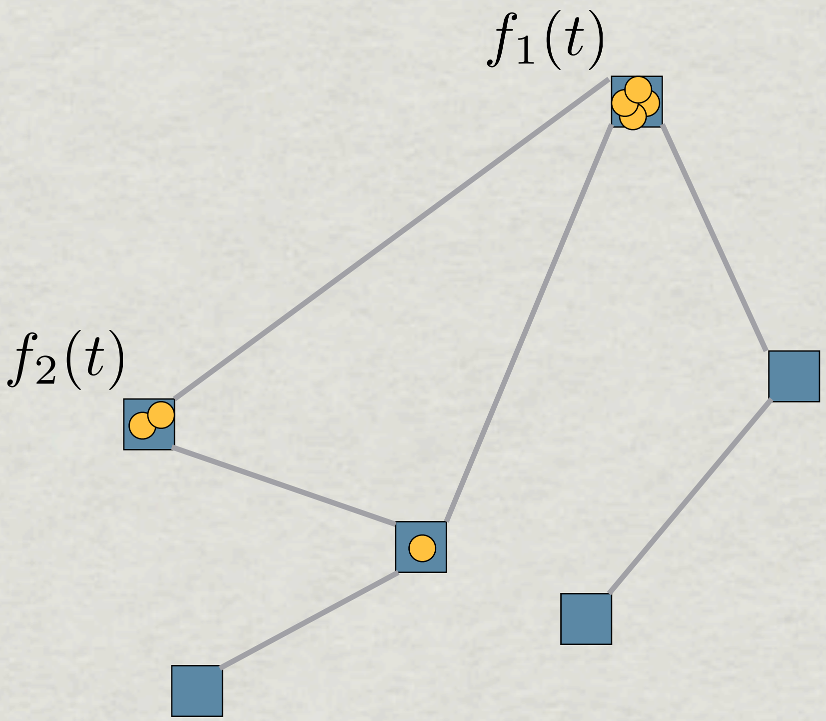


- $N(t)$ walkers
- $f_i(t) = \sum_{n=1}^{N(t)} V_{i,n}(t)$
- $\langle f_i \rangle = \langle N \rangle \langle V_{n,i} \rangle = \langle N \rangle p_i \propto k_i$

$$\sigma_i^2 = Np_i + \left[\frac{\Sigma_N}{\langle N \rangle}\right]^2 (Np_i)^2$$

$\alpha \to 1$

Random walks



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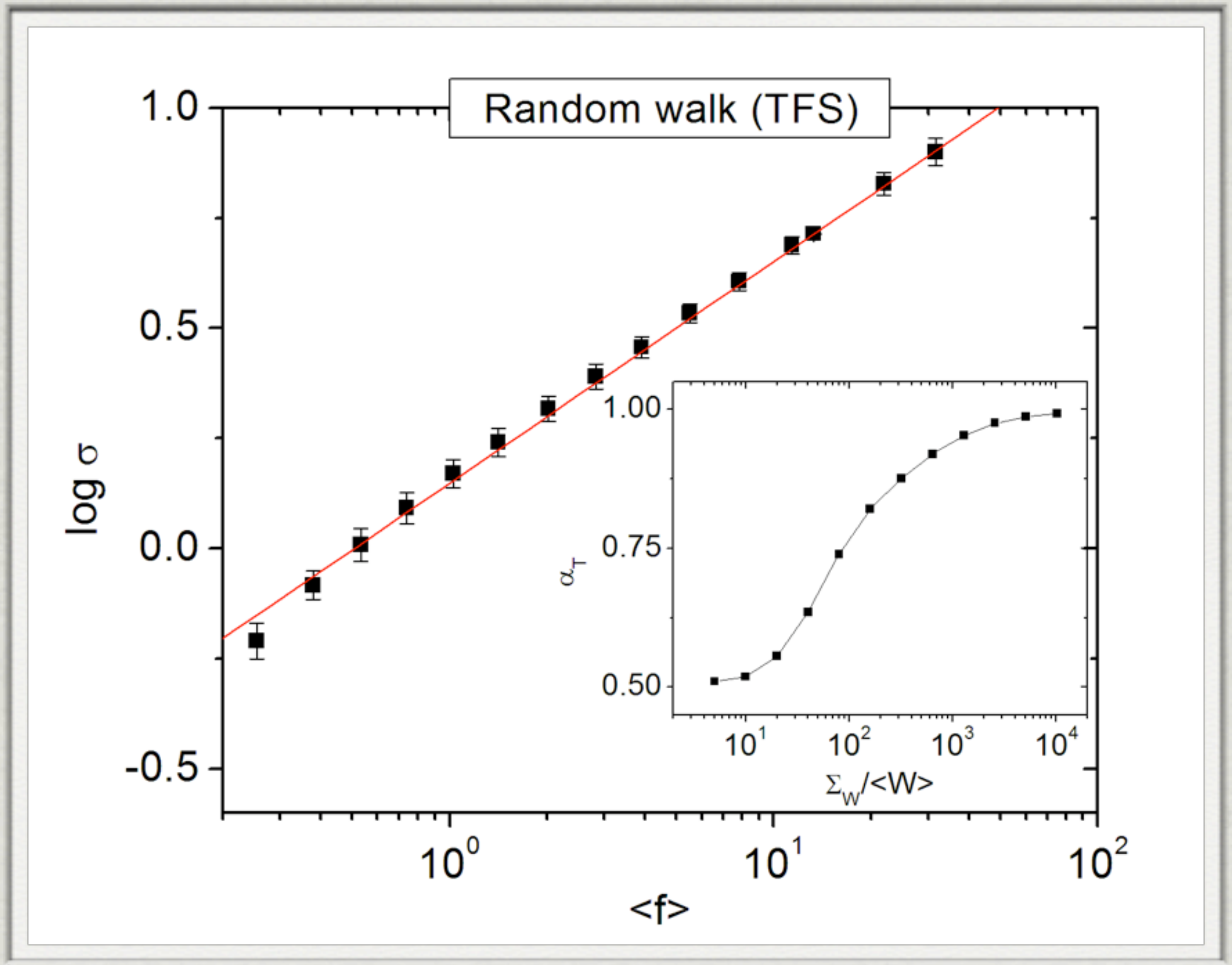
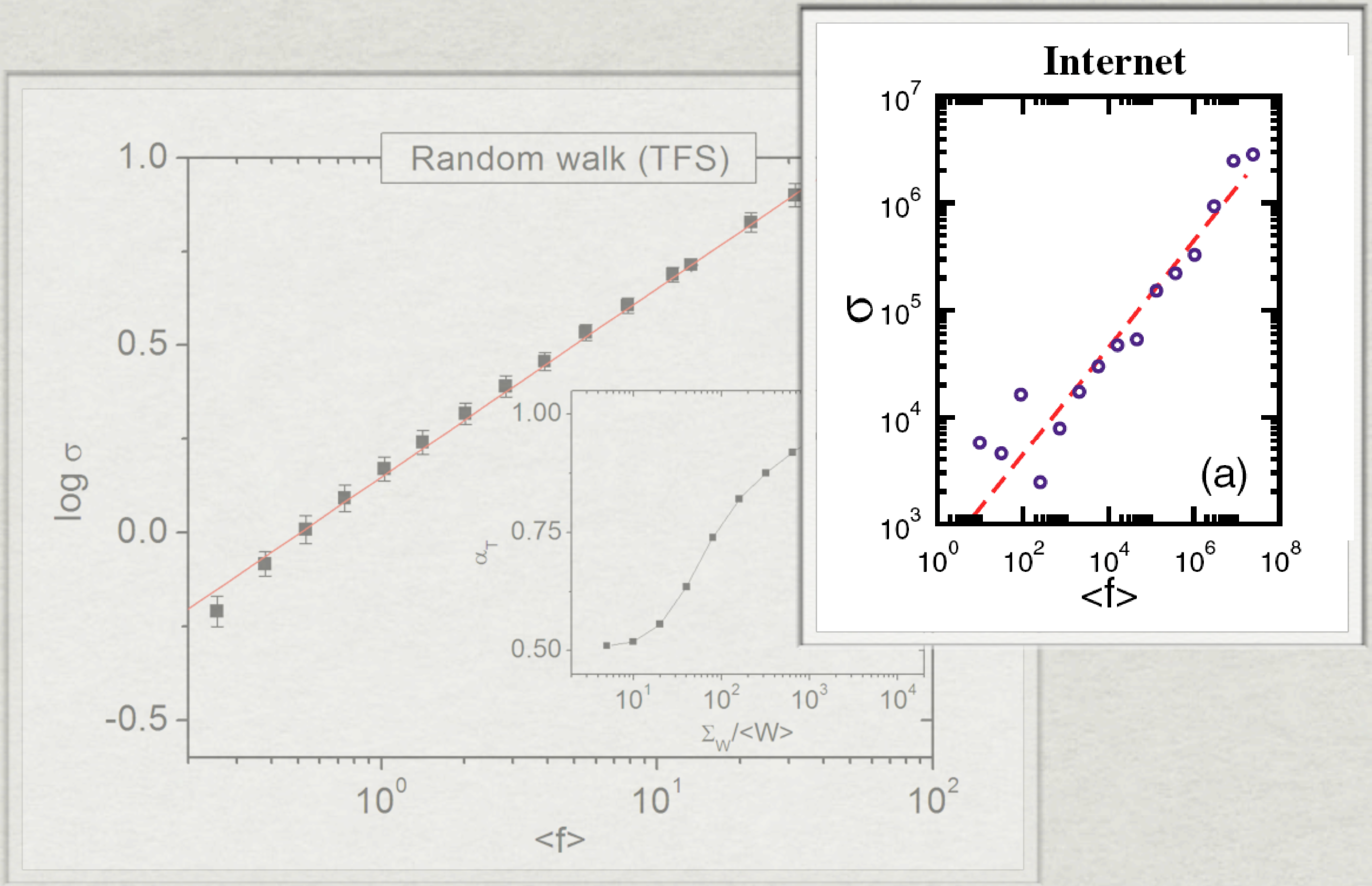
$\alpha \rightarrow 1$

Random walks



Random walks



Classification by α

- $\alpha = 1/2$: central limit theorem
- $\alpha = 1$: strongly driven system
- Universality classes?
- Any value between the two is a crossover?

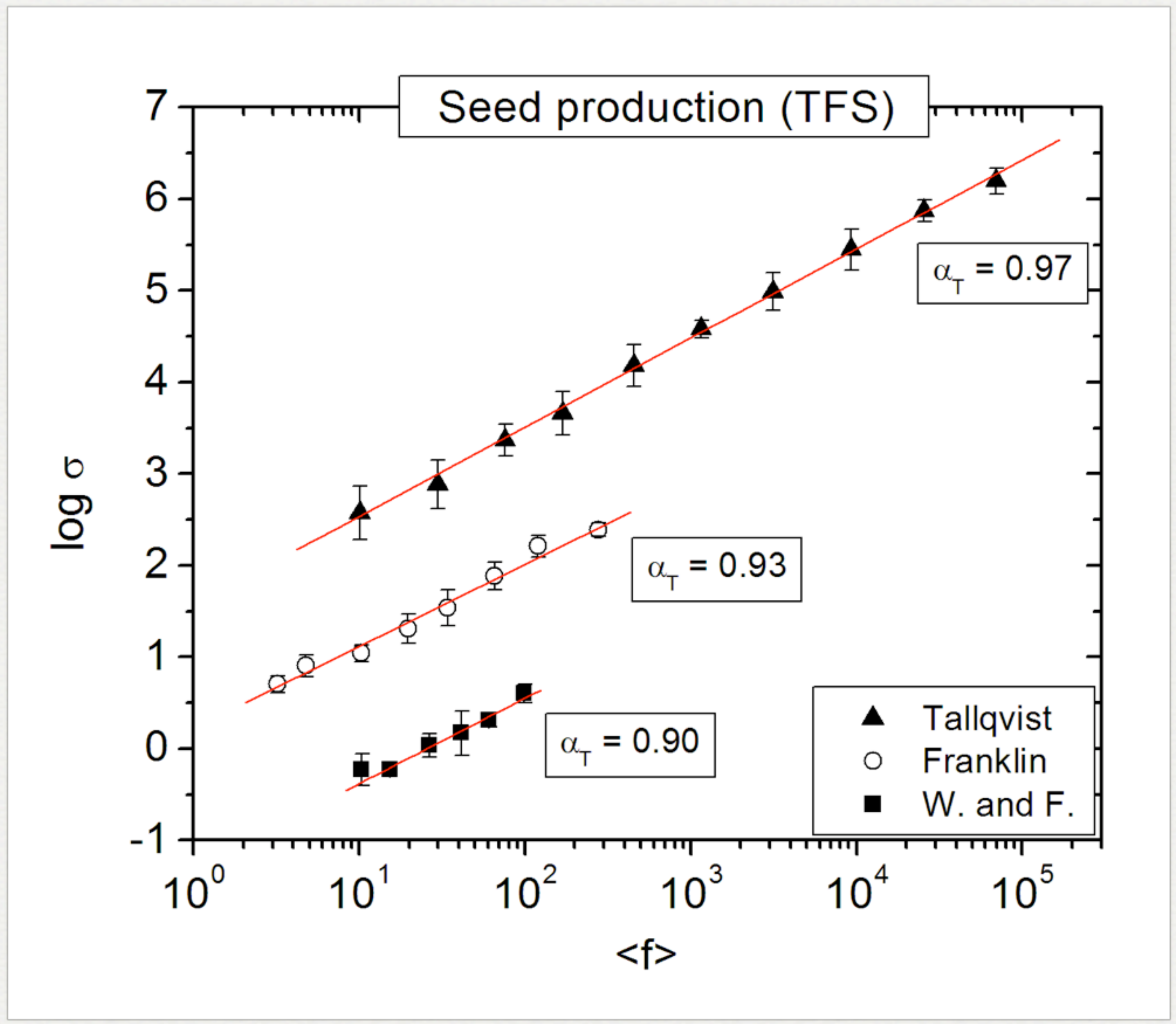
$$\sigma_i \propto \langle f_i \rangle^\alpha$$

Forests

- Consider a forest i of N_i trees
- Tree n produces $V_{n,i}(t)$ seeds in year t
- The total seed production of year t

$$f_i(t) = \sum_{n=1}^{N_i} V_{n,i}(t)$$

Forests

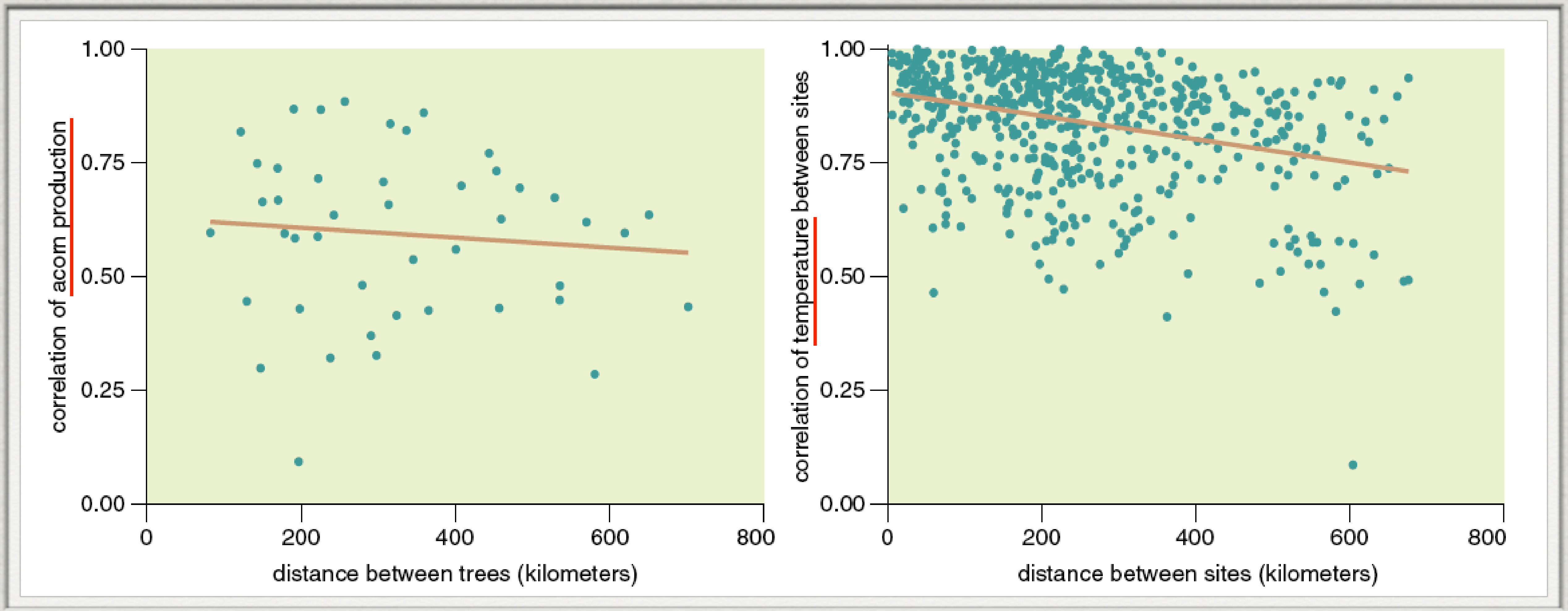


Seed production (TFS)
log σ
<f>
α_T = 0.97
α_T = 0.93
α_T = 0.90
Tallqvist
Franklin
W. and F.

Masting

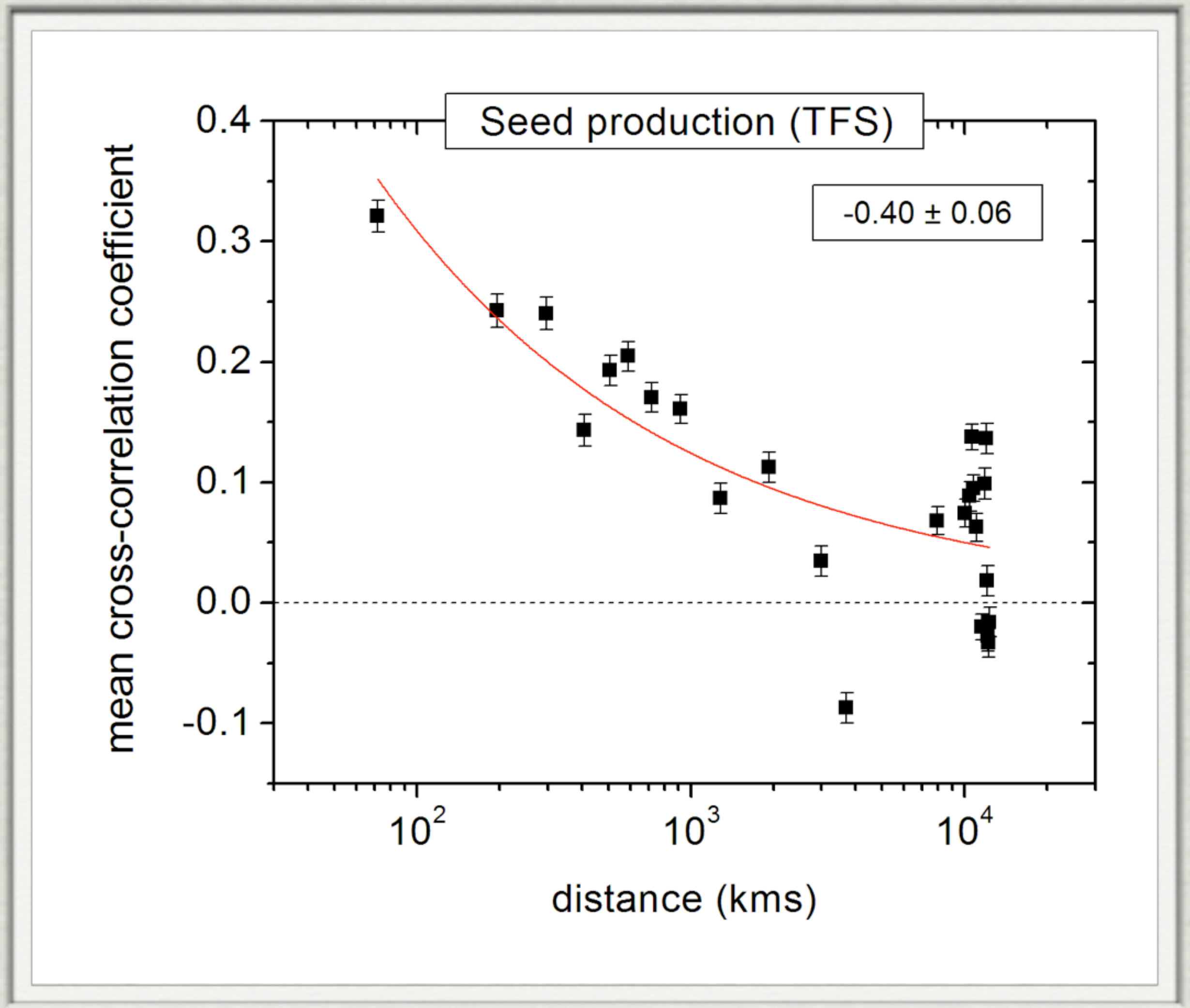
$$f_i(t) = \sum_{n=1}^{N_i} V_{n,i}(t)$$

Masting



1.00
0.75
0.50
0.25
0.00
0
200
400
600
800
correlation of acorn production
distance between trees (kilometers)
correlation of temperature between sites
distance between sites (kilometers)

Forests

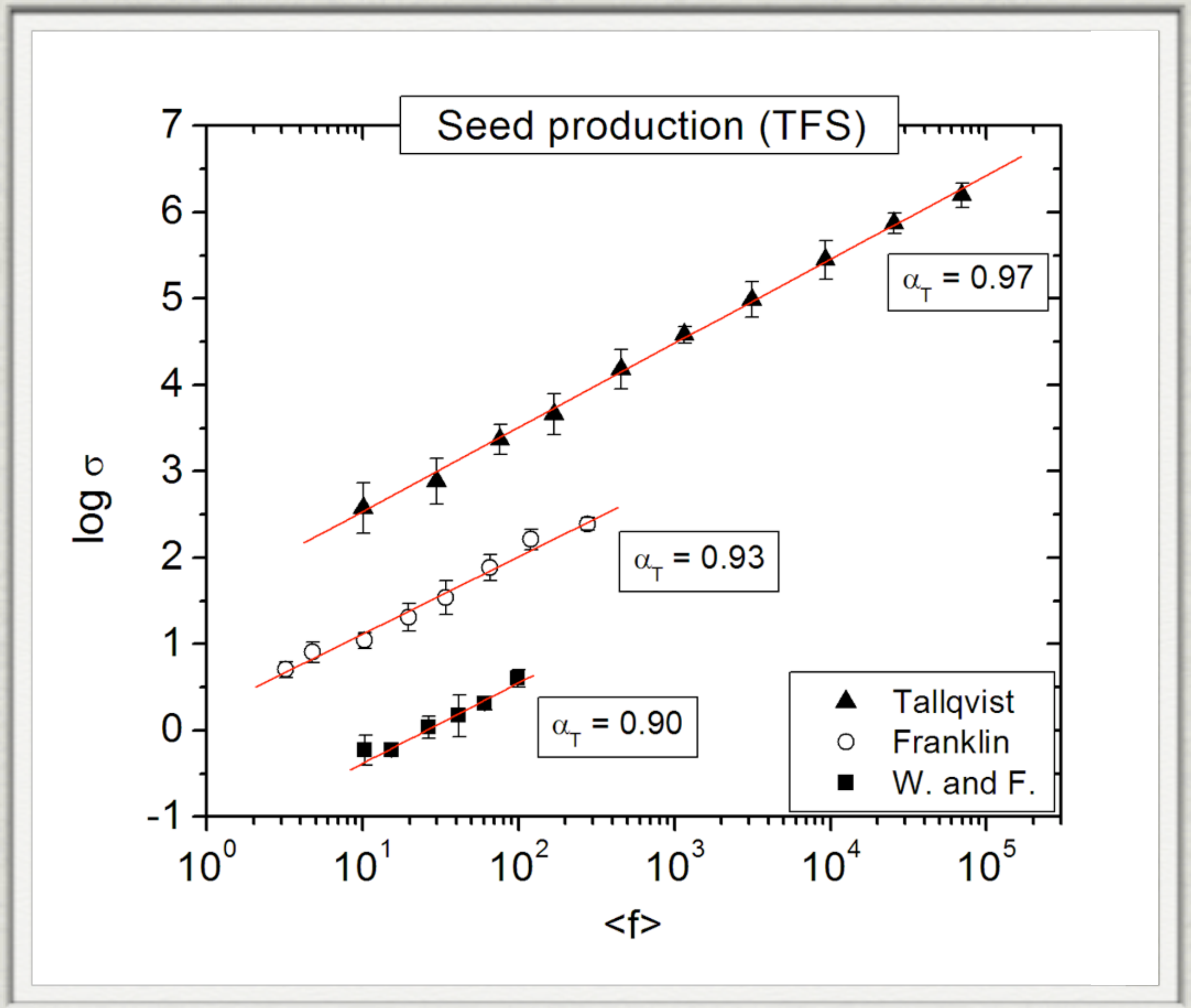


$$H_V = 1 - \frac{0.4}{2} = 0.8$$

$$\sigma_i^2 = \Sigma_{Vi}^2 \left\langle N_i^{2H_{Vi}} \right\rangle$$

$$\alpha = H_V$$

Forests



$$H_V = 1 - \frac{0.4}{2} = 0.8$$

$$\sigma_i^2 = \Sigma_{V\,i}^2 \left\langle N_i^{2H_{V\,i}} \right\rangle$$

$$\alpha = H_V$$

Forests

- Synchronization phase transition
- Satake-Iwasa model

$$\alpha = H_V$$

Animals?

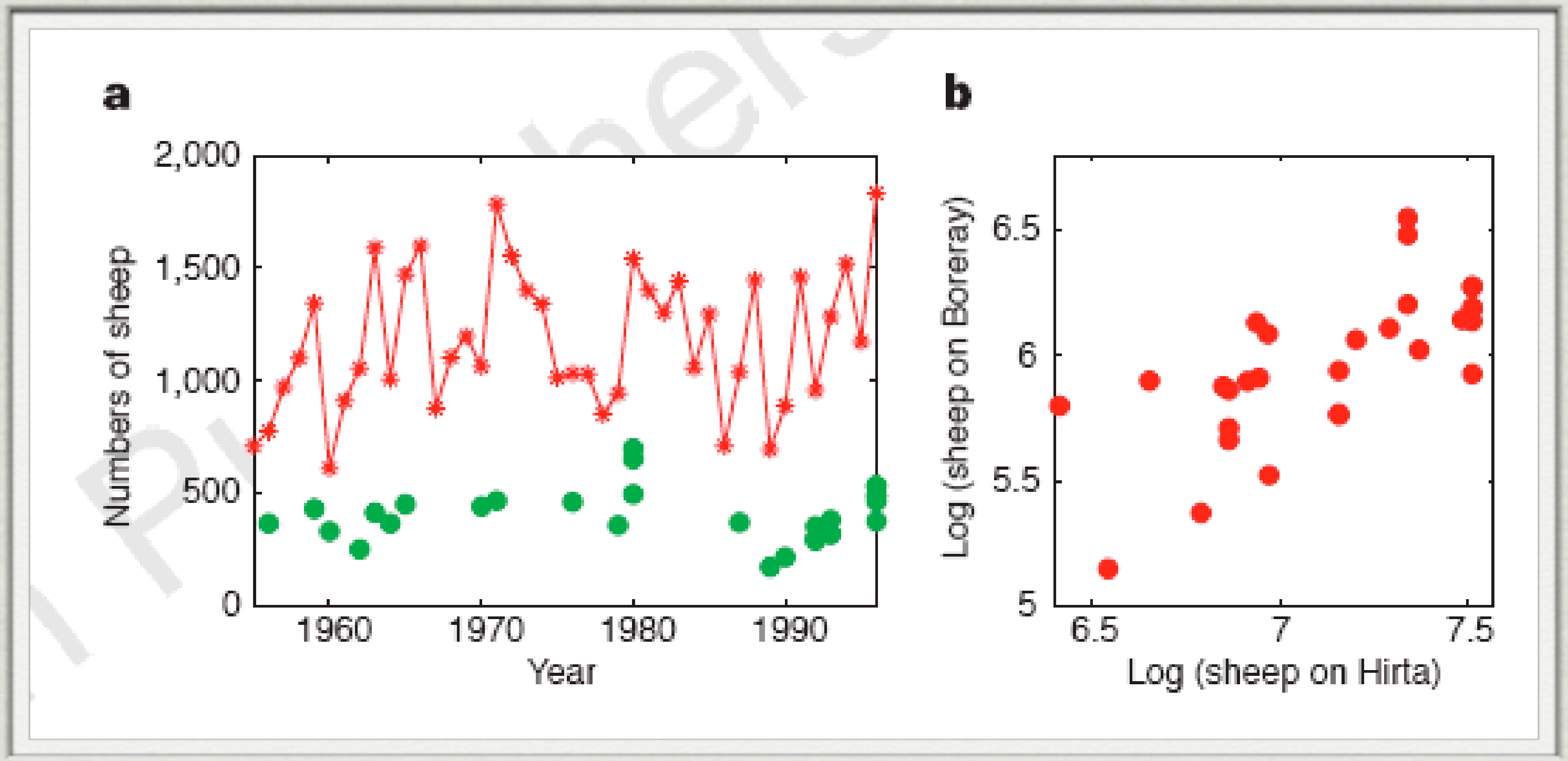
Animals?

letters to nature

Noise and determinism in synchronized sheep dynamics

B. T. Grenfell*, K. Wilson†, B. F. Finkenstädt*, T. N. Coulson‡, S. Murray§, S. D. Albon‖, J. M. Pemberton¶, T. H. Clutton-Brock* & M. J. Crawley#

Animals?



a
b
2,000
1,500
1,000
500
0
Numbers of sheep
1960
1970
1980
1990
Year
6.5
6
5.5
5
Log (sheep on Boreray)
6.5
7
7.5
Log (sheep on Hirta)

Classification by α

- α = 1/2: central limit theorem
- α = 1: strongly driven system
- 1/2 < α < 1: sums of correlated random variables

$$\sigma_i \propto \langle f_i \rangle^\alpha$$

Coin flipping

Coin flipping

- mean: 1/2, 1, 3/2, 2
- variance: 1/4, 1/2, 3/4, 1

$$\alpha = 1/2$$

Coin flipping

$$\alpha = 1$$

Coin flipping

$$\alpha = 3/4$$

Classification by α

- $\alpha = 1/2$: central limit theorem
- $\alpha = 1$: strongly driven system
- $1/2 < \alpha < 1$: sums of correlated random variables
- $1/2 < \alpha < 1$: "coin flipping"

$$\sigma_i \propto \langle f_i \rangle^{\alpha}$$

Time window dependence

$$\sigma_i \propto \langle f_i \rangle^{\alpha(\Delta t)}$$

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$$\sigma_i(\Delta t) = \left\langle \left[f_i^{\Delta t}(t) - \langle f_i^{\Delta t}(t) \rangle \right]^2 \right\rangle^{1/2} \propto \Delta t^{H_i}$$

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$$\Delta t^{H_i} \propto \langle f_i \rangle^{\alpha(\Delta t)}$$

$$\frac{dH_i}{d(\log \langle f_i \rangle)} \sim \frac{d\alpha(\Delta t)}{d(\log \Delta t)} \sim \gamma$$

Time window dependence

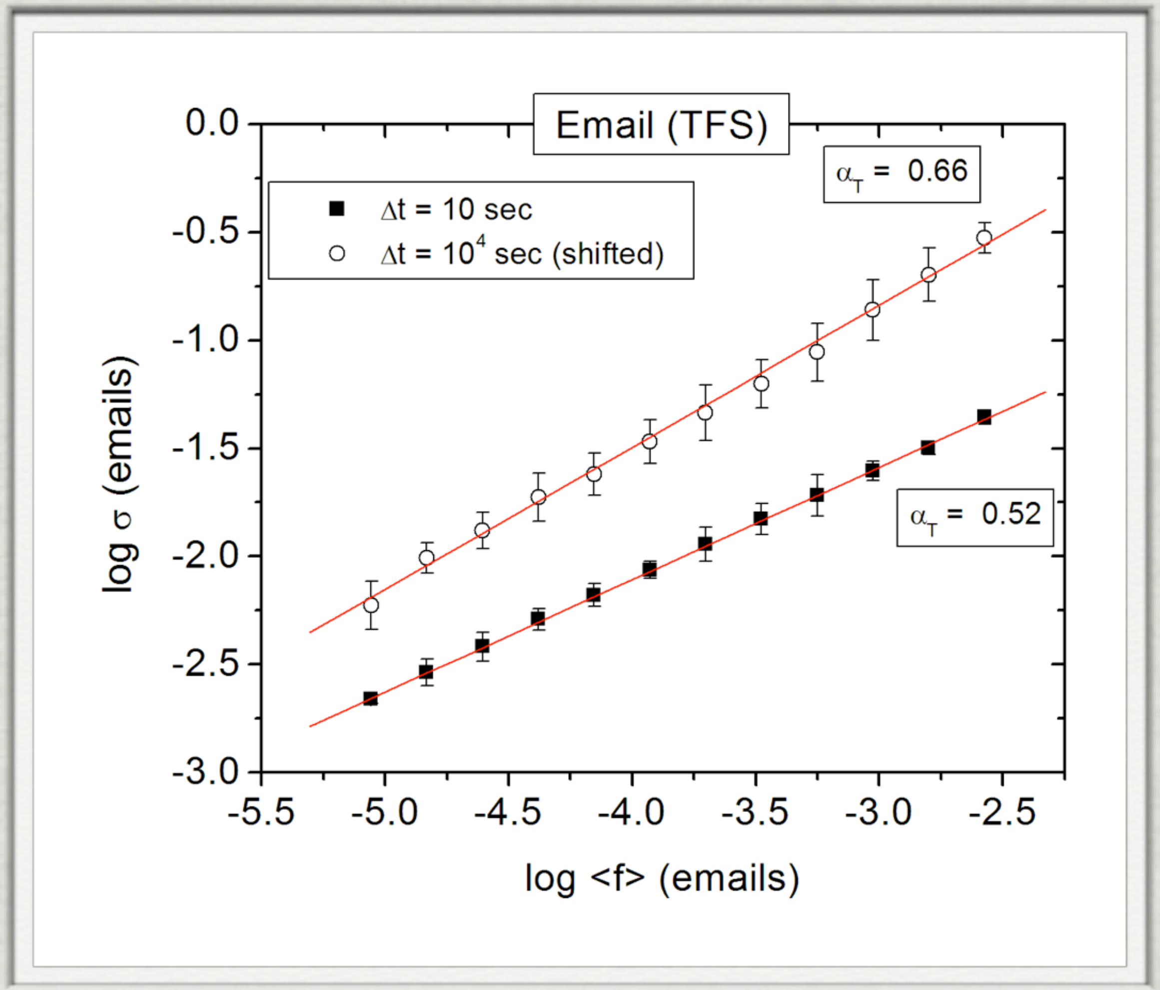
$$\sigma_i \propto \langle f_i \rangle^{\alpha(\Delta t)}$$

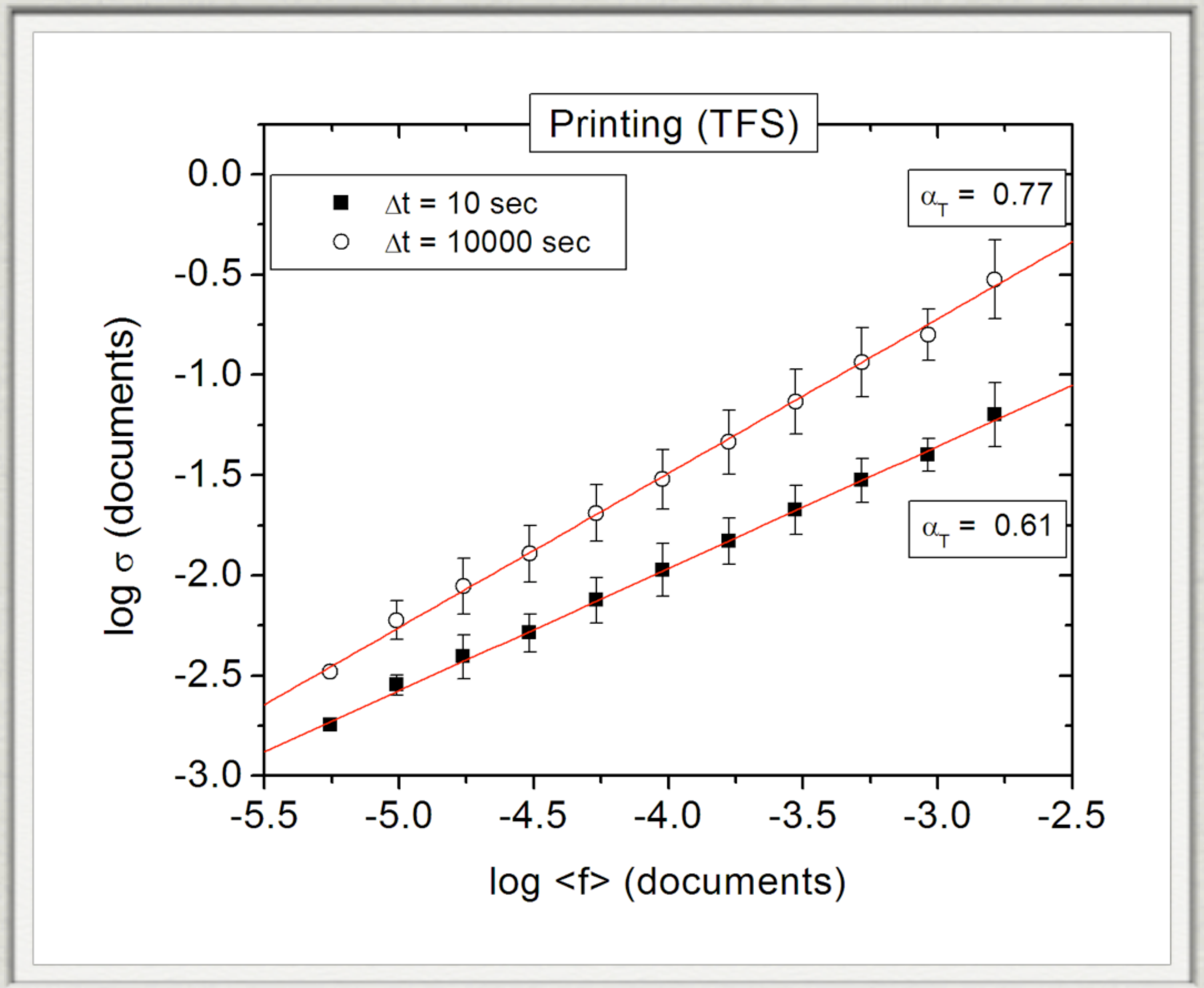
$$\sigma_i(\Delta t) = \left\langle \left[f_i^{\Delta t}(t) - \langle f_i^{\Delta t}(t) \rangle \right]^2 \right\rangle^{1/2} \propto \Delta t^{H_i}$$

$$H_i = H^* + \gamma \log \langle f_i \rangle$$

$$\alpha(\Delta t) = \alpha^* + \gamma \log \Delta t$$

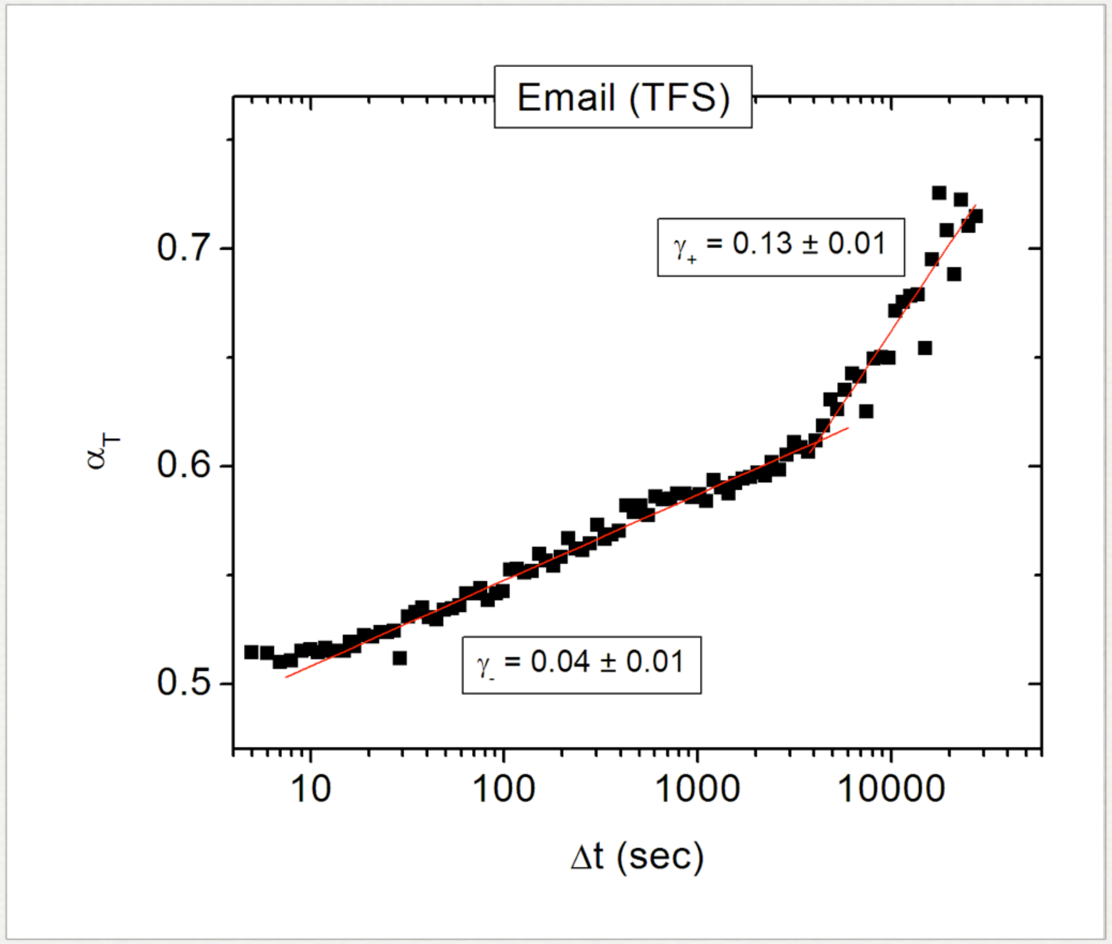
Human dynamics

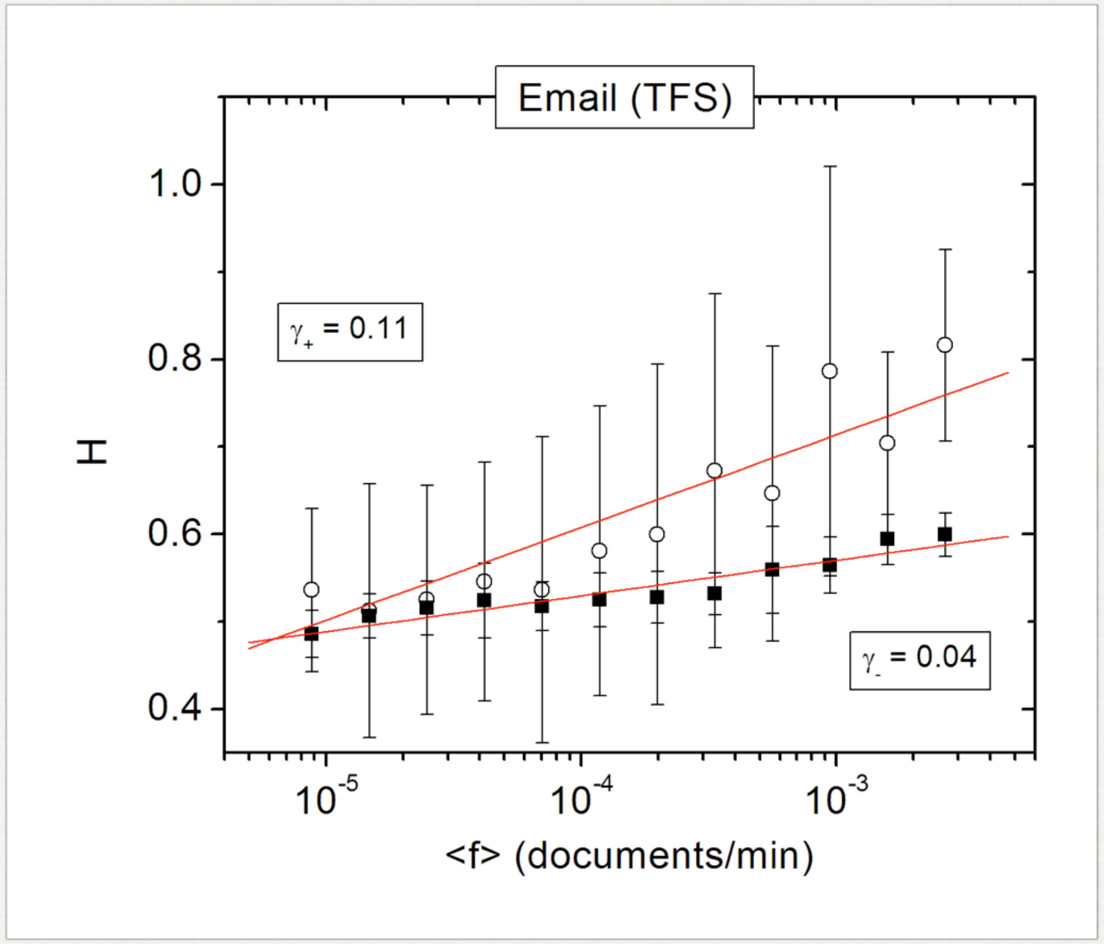




$$\sigma_i \propto \langle f_i \rangle^{\alpha(\Delta t)}$$

Human dynamics





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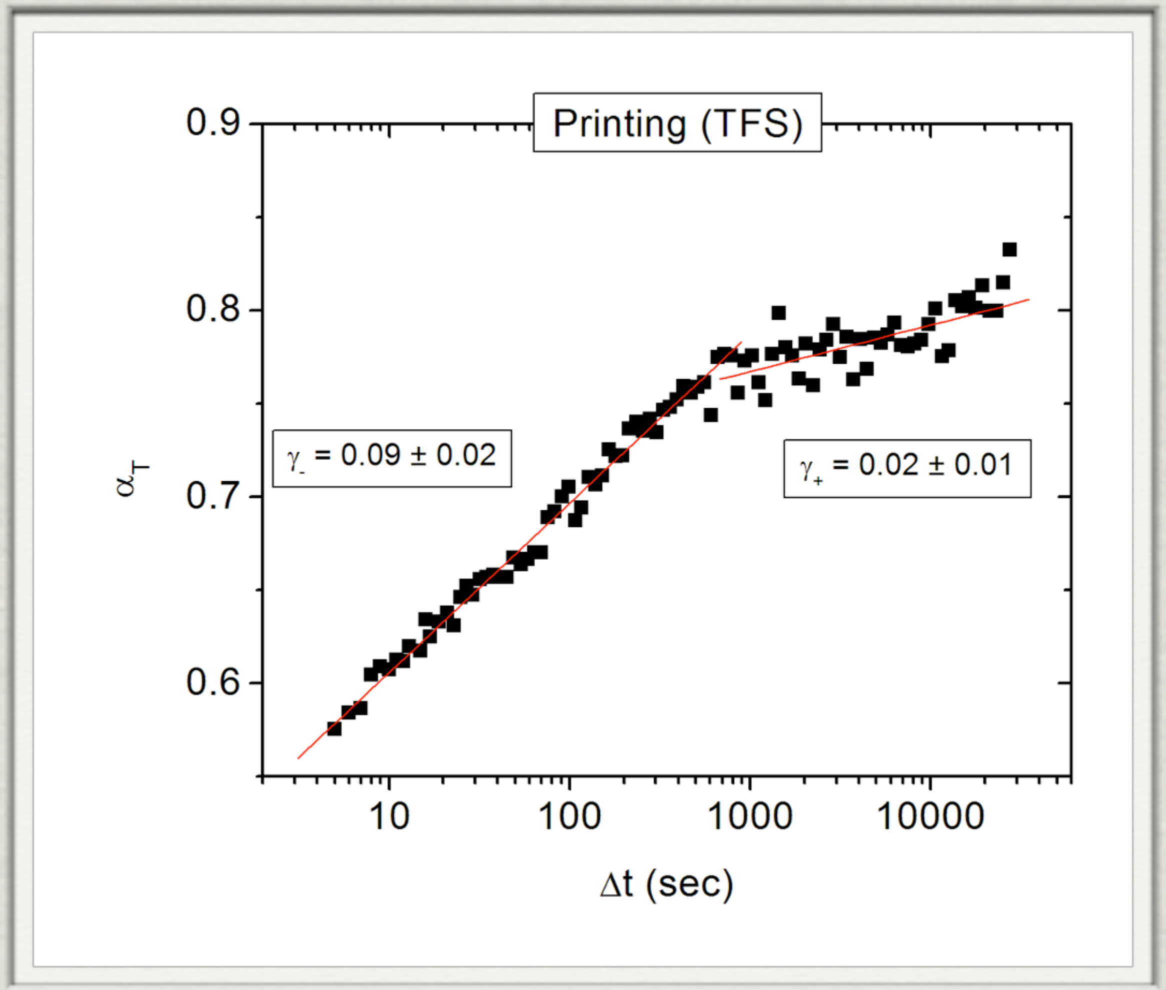
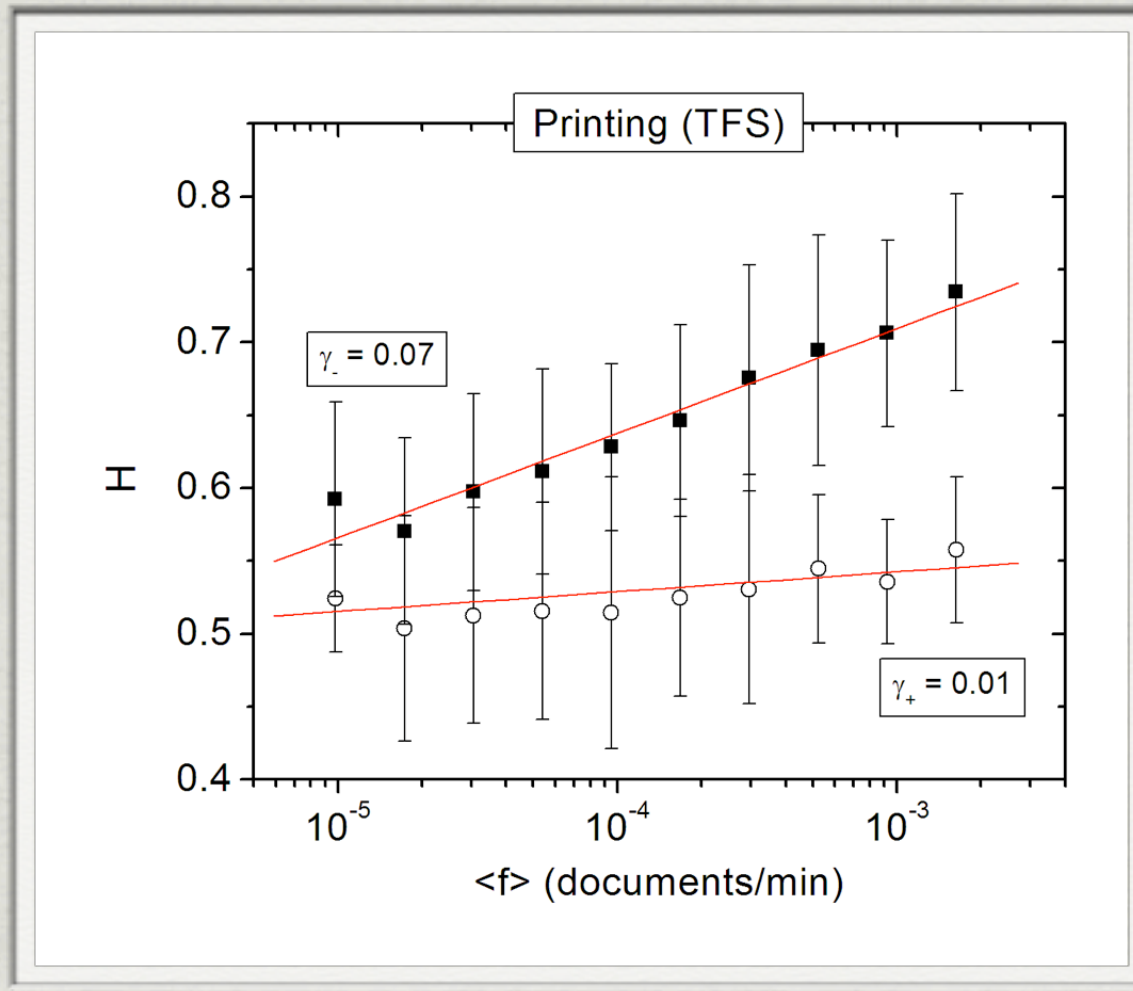
$$H_i = H^* + \gamma \log \langle f_i \rangle$$

Human dynamics





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Time window dependence

- FS enforces a logarithmic relationship on correlation strength → only the order of magnitude matters!

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Time window dependence

- FS enforces a logarithmic relationship on correlation strength → only the order of magnitude matters!
- order of magnitude matters → non-universality
- α can take any value depending on the time resolution
- one must map a range in Δt

Conclusions

- Fluctuation scaling: in any field with positive, additive quantities
- The exponent α can be used to gain hints about dynamics
- Empirical observation of limit theorems?
- Hurst exponents change logarithmically with size?

References

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